STUDY OF EMITTANCE GROWTH CAUSED BY SPACE CHARGE AND LATTICE INDUCED RESONANCES

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Abstract

Emittance growth and beam loss in high intensity circular proton accelerators are one of the most serious issues which limit their performance. The emittance growth is caused by linear and nonlinear resonances of betatron/synchrotron oscillation due to lattice and space charge nonlinear force. The resonances are induced by errors in many cases. The space charge effects have been studied by computer simulations. Simulations with taking into account errors at random are consuming. We should first understand which resonances are serious. Resonance strength and resonance width induced by space charge and lattice nonlinearity is discussed with integrals along a ring like the radiation integrals. Emittance growth is evaluated by model with the resonance width to understand the mechanism.

INTRODUCTION

Particles move with experience of electro-magnetic field of lattice elements and space charge. We study slow emittance growth arising in a high intensity circular proton ring. We assume that the beam distribution is quasi-static, and where Hamiltonian is represented by action variables $J$ with integrals along a ring like the radiation integrals. Emission growth is evaluated by model with the resonance width and tune slope for amplitude as follows \[1\], by three parts for (1) linear betatron/synchrotron motion (2) nonlinear component of the lattice magnets ($U_{nl}$) and (3) space charge potential ($U$).

$$H = \mu J + U_{nl} + U_{sc}.$$  \hfill (1)

where Hamiltonian is represented by action variables $J$ and $\phi$, which are Courant-Snyder invariant ($W = 2J$) and betatron phase, respectively. Hamiltonian is expanded by Fourier series,

$$H = \mu J + U_{00} + \sum_{m_x,m_y \neq 0} U_{m_x,m_y}(J) \exp(-im_x \phi_x - im_y \phi_y).$$  \hfill (2)

where Phase space structure near resonances are characterized by the resonance width. It is determined by their strength and tune slope for amplitude as follows \[1\],

$$\Delta J_x = 2 \sqrt{\frac{U_{m_x,m_y}}{\Lambda}} \quad \Lambda = \frac{\partial^2 U_{00}}{\partial J_x^2}.$$  \hfill (3)

EVALUATION OF RESONANCE WIDTH

Resonances Due to Space Charge Force

We first discuss the space charge potential $U_{sc}$ \[2\]. Beam distribution is assumed to be Gaussian in transverse determined by emittance and Twiss parameters. $U$ contains linear component, which gives a tune shift and Twiss parameter distortion. Twiss distortion is given by solving an envelope equation including linear space charge force self-consistently.

$$U_{sc} = \int ds' U_{sc}(s') = \frac{\lambda_p r_p}{\beta y^3} \int ds'$$

$$\int_0^\infty 1 - \exp \left(-\frac{\beta_x(s')X(s,s') - \beta_y(s')Y(s,s')}{2\sigma_x^2 + u} \right)du$$

where $X$ and $Y$ are normalized betatron coordinates at $s'$ as

$$X(s,s') = \sqrt{2J_x} \cos(\varphi_x(s') + \phi_x(s))$$

$$Y(s,s') = \sqrt{2J_x} \cos(\varphi_y(s') + \phi_y(s)).$$  \hfill (5)

where $\varphi_{x,y}(s')$ is the betatron phase difference between $s$ and $s'$.

The Fourier component, which correspond to resonance strength, is given by

$$U_{m_x,m_y}(J_x, J_y) = \frac{\lambda_p r_p}{\beta y^3} \int ds \int_0^\infty du \left[ \delta_{m_x,0} \delta_{m_y,0} - \exp(w_x - w_y)(-1)^{(m_x + m_y)/2} I_{m_x/2}(w_x) I_{m_y/2}(w_y) e^{-im_x \varphi_x - im_y \varphi_y} \right].$$  \hfill (6)

The tune slope $\partial U_{00}/\partial J_x^2$ in Eq.(3) induced by space charge potential is obtained as follows. The tune slope is evaluated by $U_{00}(J_x, J_y)$ in Eq.(7).

$$U_{00}(J_x, J_y) = \frac{\lambda_p r_p}{\beta y^3} \int ds \int_0^\infty \frac{d\eta}{\sqrt{2 + \eta} \sqrt{2r_{xx} + \eta}}$$

$$(1 - e^{-w_x - w_y} I_0(w_x) I_0(w_y)).$$  \hfill (7)

where $r_{yy} = \sigma_{yy}^2/\sigma_{y}^2$ and

$$w_x = \frac{\beta_x J_x/\sigma_x^2}{2 + \eta}, \quad w_y = \frac{\beta_y J_y/\sigma_y^2}{2 + \eta/r_{yy}}.$$  \hfill (8)

$$\frac{\partial}{\partial J_x} = \frac{\beta_x J_x/\sigma_x^2}{2 + \eta} \frac{\partial}{\partial w_x}, \quad \frac{\partial}{\partial J_y} = \frac{\beta_y J_y/\sigma_y^2}{2r_{yy} + \eta} \frac{\partial}{\partial w_y}.$$  \hfill (9)
The tune shift is given by derivative of $U_{00}$ for $J_{xy}$ as follows,

$$2\pi \Delta \nu_x = -\frac{\partial U_{00}}{\partial J_x}$$

where $I_0(x)' = I_1(x)$, $I_0(x)' = (I_0(x) + I_2(x))/2$ are used. Similar formula are given for $2\pi \Delta \nu_y$.

The tune slope is given by second derivative of $U_{00}$ as follows,

$$\frac{\partial^2 U_{00}}{\partial J_x^2} = -2\pi \frac{\partial \nu_x}{\partial J_x}$$

Figure 1 shows tune spread ($\Delta \nu_{x,y}(J_x, J_y)$), slope ($\partial^2 U_{00}/\partial J_x^2$), 4-th order resonance strength ($U_{4,0}$) and its width due to space charge force for J-PARC MR. The resonance width is visible size, $0.2\varepsilon$, when $J_R = \varepsilon$.

Figure 2 shows the tune shift and slope. Typical tune slope is $\partial^2 U_{00}/\partial x^2 = 1000 \sim 3000$. This value is similar for $U_{x,y,0}$ at $J_x = 3\varepsilon$, namely tune slope of space charge is dominant for that of lattice nonlinearity at $J < 9\varepsilon(3\sigma)$, vice versa.

Resonance strength due to lattice nonlinearity is obtained by the one turn map. Table 1 shows the resonance strength $U_{m,m,y}(J)$, up to 4-th.

Table 1: $U_{m,m,y}(J)$ for lattice nonlinearity. $U$‘s are evaluated at $J$ 3rd and 4-th column. The suffix, B0, B and BR means lattices without errors, lattice with measured beta and measured beta and coupling [3].

Figure 2: Tune spread, $\partial^2 U_{00}/\partial x^2$ induced by lattice nonlinearity.
TOY MODEL WITH THE TUNE SLOPE AND RESONANCE STRENGTH

We study emittance growth for an accelerator model with a given tune slope and resonance strength. This is an example of Hamiltonian,

\[ H = \mu_0 J + \left( J + e^{-2aJ} \right) + bJ \cos m\phi. \]  

(13)

The tune shift is given by

\[ \mu = \frac{\partial H}{\partial J} = \mu_0 + (1 - e^{-2aJ}). \]  

(14)

For small amplitude tune shift \( 2aJ \), where \( a > 0 \). The tune slope is given by

\[ \frac{\partial^2 H}{\partial J^2} = 2ae^{-2aJ}. \]  

(15)

Half width of the resonance is expressed by

\[ \Delta J = \sqrt{\frac{2b J_R}{ae^{-2aJ_n}}}. \]  

(16)

Symplectic integration is performed by \( H(J,\phi) \) as follows,

\[ J_{n+1} = \frac{J_n}{1 - bm \sin m\phi_n}, \]  

\[ \phi_{n+1} = \phi_n + \mu + (e^{-2aJ_{n+1}} - 1) + b \cos m\phi_n, \]  

where \( J_n \) and \( \phi_n \) are those of \( n \)-th turn.

We study two cases of parameters,

- \( a = 0.5, b = 0.002, m = 4, \mu = 2\pi \times 0.203 \)
- \( a = 0.5, b = 0.0002, m = 4, \mu = 2\pi \times 0.203 \)

The resonance widths are given as (1) \( \Delta J = 0.07 \) and (2) \( \Delta J = 0.02 \). The betatron amplitude, where the resonance hits, is \( J_R = 0.38 \).

The model is tracked using the two sets of parameters. Figure 3 shows phase space trajectories. 4-the order resonance is seen, and their position \( J_R \) and widths agree with the formula, Eqs.(14) and (16).

\[ \text{Tune spread area modulates due to synchrotron oscillation.} \]

To study the effect, the strength of tune shift term \( a \) is made a modulation as

\[ a = \frac{a_0}{2} (1 + \cos 2\pi \nu_s n). \]  

(18)

The resonant amplitude moves to larger amplitude for small \( a \). The model does not match to space charge force in this point. This model should be improved in the future. Figure 4 shows phase space plot taking into account of the effective synchrotron motion. Chaotic area drastically increases due to the synchrotron motion. Figure 5 shows the emittance growth of the model with Eqs.(17) and (18). We can see the emittance growth depending on the resonance width.

\[ \text{SUMMARY} \]

Tune slope and resonance strength induced by lattice and space charge nonlinear force were evaluated by integrals along ring. The resonance width which characterizes emittance growth is estimated by them. A simple model with the resonance information is examined to study emittance growth. Synchrotron motion is taken into account of changing space charge tune shift for \( z \). Enhancement of the emittance growth was evaluated.

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\[ \text{REFERENCES} \]